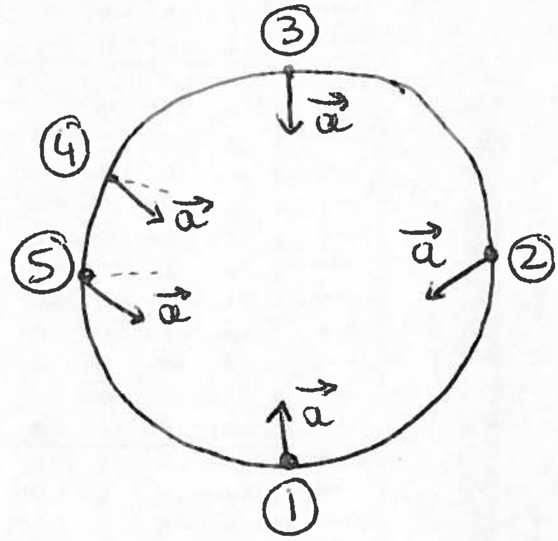
Loop the loop. (Example 2)

a)



Recall that

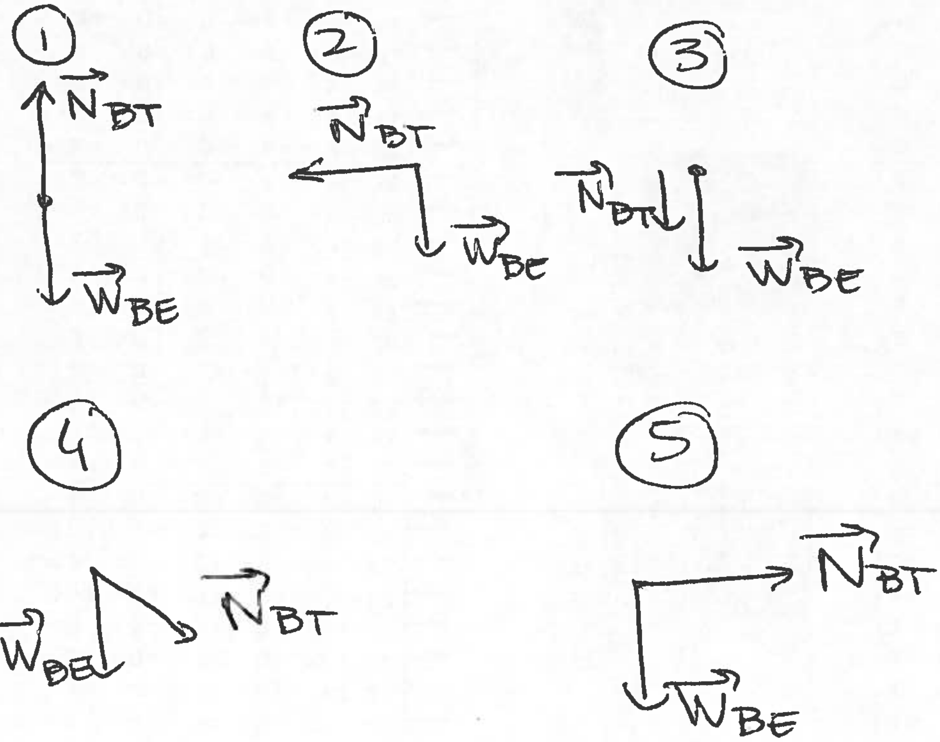
$\vec{a} \parallel \Delta\vec{v}$

b)

B: ball

T: track

E: Earth



$\vec{N}_{BT}$ is always directed to the center of the track.

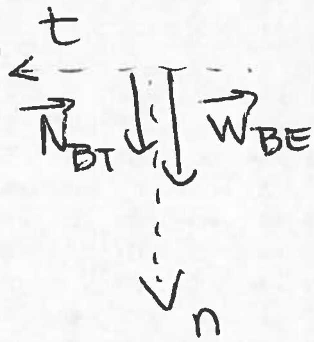
c)

$$a_t = \frac{dv}{dt}, \quad a_n = \frac{v^2}{r}$$

a_n: centripetal

d) Find V_{min} at the top first.

From ③



$$\vec{N}_{BT} + \vec{W}_{BE} = m\vec{a}$$

$$t: \quad 0 = ma_t \Rightarrow a_t = 0$$

$$n: \quad N_{BT} + mg = m\frac{v^2}{r}$$

$$\rightarrow N_{BT} = m\left(\frac{v^2}{r} - g\right)$$

the ball stays on the track if

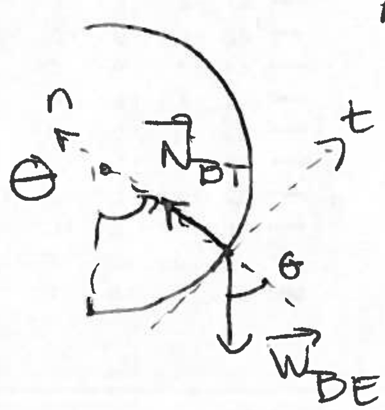
$$N_{BT} \geq 0$$

$$m\left(\frac{v^2}{r} - g\right) \geq 0$$

$$v \geq \sqrt{gr}$$

so $V_{min}^{top} = \sqrt{gr}$

To find V_{min}^{bottom}, analyze the motion of the ball as it climbs up the track.



At position θ,

$$\vec{N}_{BT} + \vec{W}_{BE} = m\vec{a}$$

$$t: \quad -mg\sin\theta = m\frac{dv}{dt}$$

$$\rightarrow \frac{dv}{dt} = -g\sin\theta$$

Use $V = r\omega = r\dfrac{d\theta}{dt}$

$$\frac{dv}{dt} = r\frac{d^2\theta}{dt^2}$$

So we solve:

$$r\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$\text{or: } \frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta.$$

Trick: multiply both sides by $\dfrac{d\theta}{dt}$ and apply the chain rule.

$$\underbrace{\frac{d^2\theta}{dt^2}\cdot\frac{d\theta}{dt}}_{\frac{d}{dt}\left(\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2\right)} = -\frac{g}{r}\underbrace{(\sin\theta)\frac{d\theta}{dt}}_{\frac{d}{dt}(-\cos\theta)}.$$

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2\right) = -\frac{g}{r}\frac{d}{dt}\left(-\cos\theta\right)$$

Use again: $V = r\dfrac{d\theta}{dt} \Rightarrow \dfrac{d\theta}{dt} = \dfrac{V}{r}.$

$$\frac{d}{dt}\left(\frac{1}{2}\frac{v^2}{r^2}\right) = \frac{g}{r}\frac{d}{dt}(\cos\theta)$$

Integrate between $t = t_{bot}$ and $t = t_{top}$,

$$\frac{1}{2r^2}\left(v_{top}^2 - V_{bot}^2\right) = \frac{g}{r}\left(\underset{\substack{\uparrow\\ \text{top}}}{\cos 180^\circ} - \underset{\substack{\uparrow\\ \text{bot}}}{\cos 0^\circ}\right)$$

$$v_{top}^2 - v_{bot}^2 = 2gr(-1-1) = -4gr$$

so

$$v_{bot}^2 = v_{top}^2 + 4gr$$

Since $v_{top} \geq \sqrt{gr}$

$$v_{bot}^2 \geq gr + 4gr$$

$$v_{bot} \geq \sqrt{5gr}$$

The ball stays on the track all around the loop if $\boxed{v_{bot} \geq v_{min,bot} = \sqrt{5gr}}$